

The hanging thin rod: A singularly perturbed eigenvalue problem

Yossi Farjoun[†]David G. Schaeffer[‡]

August 12, 2010

Abstract

We study the vibrations of a hanging thin flexible rod, in which the dominant restoring force in most of the domain is tension due to the weight of the rod, while bending elasticity plays a small but non-negligible role. We consider a linearized description, which we may reduce to an eigenvalue problem. We solve the resulting singularly perturbed problem asymptotically up to the first modification of the eigenvalue. On the way, we illustrate several important problem-solving techniques: modeling, nondimensionalization, scaling, and especially use of asymptotic series.

1 Introduction

Linear transverse vibrations of a uniformly stretched string are modeled with the wave equation

$$\rho w_{tt} - Tw_{yy} = 0, \quad (1)$$

where $w(y, t)$ is the displacement of the string, ρ the mass per unit length, and T the tension. In (1) the string is assumed perfectly flexible; if resistance to bending is not negligible (e.g., for a rod), then the equation requires an additional, fourth-order, term,

$$\rho w_{tt} - Tw_{yy} + EIw_{yyyy} = 0, \quad (2)$$

where E is Young's modulus and I is the cross-sectional moment of area. If the tension depends on position y , then the middle term of (2) is replaced by a variable-coefficient operator in divergence form; in particular, if the rod is vertical so that the tension at a point y results solely from the weight of the rod below it, located say in $[0, y]$, then the tension is $T(y) = \rho gy$ and (2) is modified to

$$\rho w_{tt} - \rho g(yw_y)_y + EIw_{yyyy} = 0, \quad (3)$$

where g is the acceleration of gravity. In this paper we study (3) in the asymptotic limit where the bending forces are small compared to tension, using singular perturbations*.

[†]G. Millán Institute of Fluid Dynamics, Nanoscience, and Industrial Mathematics
Universidad Carlos III de Madrid, Avenida de la Universidad 30, Leganés, Spain, 28911
Corresponding Author: yfarjoun@ing.uc3m.es

[‡]Duke University, Department of Mathematics and Center for Nonlinear and Complex Systems,
Durham NC 27708-0320, USA

*We consider only deflections in one transverse direction. In the linear approximation assumed in (3), vibrations in the two transverse directions are independent.

Our interest in this problem originated from Manela and Howe’s paper [MH09] simulating the waving of a flag in the wind. The displacement of the flag was represented as an expansion in terms of eigenfunctions of the second-order linear operator

$$w \mapsto \frac{\partial}{\partial y} \left(y \frac{\partial w}{\partial y} \right);$$

i.e., the part of (3) due to tension alone, neglecting bending resistance. It was found that this expansion converged very poorly, if at all. In this paper we study the effect of a small bending resistance on the eigenfunctions, without reference to wind-driven motion.

The problem of fluttering plates and rods driven by a fluid has been studied earlier in various places. In a recent paper, Argentina and Mahadevan [AM05] study the problem where the bending rigidity dominates, and in a previous paper, Dowling [Dow87] uses matched asymptotic expansions to solve a slightly different problem where the fluid-loading is taken into account. This has the consequence of moving the main singular point into the bulk which results in significantly different behavior.

2 Preliminary Analysis

2.1 Boundary Conditions

Equation (3) must be supplemented with two boundary conditions at each end of the domain, say $0 \leq y \leq L$. At the free end $y = 0$ there should be no bending moment (i.e., $EI w_{yy}(0, t) = 0$) and no force. At a point where the tension vanishes, the force is given by the negative derivative of the bending moment, so we obtain the second boundary condition $w_{yyy}(0, t) = 0$.

At $y = L$ we consider two distinct possibilities for boundary conditions: clamped and pinned. Both imply that there is no deflection at $y = L$; thus, $w(L, t) = 0$. For the second boundary condition, clamped imply zero slope (i.e., $w_y(L, t) = 0$), and pinned imply zero bending moment (i.e., $w_{yy}(L, t) = 0$).

Thus, in summary, the two possible sets of boundary conditions are

$$w_{yy}(0, t) = 0, \quad w_{yyy}(0, t) = 0, \quad w(L, t) = 0, \text{ plus } \begin{cases} \text{either} & w_y(L, t) = 0, \quad (\text{Clamped}) \\ \text{or} & w_{yy}(L, t) = 0. \quad (\text{Pinned}) \end{cases} \quad (4)$$

2.2 Non-dimensionalization and Scaling

Despite there being several parameters that control the behavior of (3), these may be reduced to a single non-dimensional “group.” Let us scale y by the length L and t by L/c , where $c = \sqrt{gL}$ specifies the order of magnitude of the speed tension waves; i.e., let

$$\tilde{y} = \frac{y}{L}, \quad \tilde{t} = \sqrt{\frac{g}{L}} t. \quad (5)$$

The reason we use the timescale of the tension waves is that we are interested in looking at the *thin rod* case, where the elasticity plays a small role in the dynamics. (We do not scale w since in a linear equation this would not change anything.)

Substituting (5) into (3) yields

$$w_{\tilde{t}\tilde{t}} - (\tilde{y} w_{\tilde{y}})_{\tilde{y}} + \varepsilon w_{\tilde{y}\tilde{y}\tilde{y}\tilde{y}} = 0, \quad \text{where } \varepsilon = \frac{EI}{\rho g L^3}. \quad (6)$$

The dimensionless constant ε compares the importance of bending elasticity ($\mathcal{O}(EI/L^2)$) with the maximum tension ($\rho g L$). Of course ε is small if L is large. Let us also examine the dependence of ε on a , the width of the rod. The second moment of area is given by

$$I = \int_C x^2 dx dz, \quad (7)$$

where C is the cross-sectional area of the rod and x is the direction of bending. This moment scales like a^4 , while ρ , mass *per unit length*, scales like a^2 . Thus, ε contains an implicit factor of $a^{2\dagger}$.

Below we omit the tildes from (6).

2.3 Separation of Variables

We look for a solution of (6) in separated form $w(y, t) = u(y) \times \Omega(t)$, and find that u must satisfy

$$\varepsilon u'''' - (yu')' = \lambda u, \quad 0 < y < 1, \quad (8)$$

where λ is the eigenvalue parameter, and

$$\Omega(t) = \Omega_0 e^{\pm i\sqrt{\lambda}t}. \quad (9)$$

Note that (8) is singular for two reasons: (i) ε multiplies the highest-order derivative in the equation and (ii) the coefficient of leading-order derivative in the reduced equation (after setting $\varepsilon = 0$),

$$-(yu')' = \lambda u, \quad (10)$$

vanishes at one end of the interval. In the remainder of the paper we solve asymptotically this *singularly perturbed* eigenvalue problem, subject to the boundary conditions

$$u''(0) = 0, \quad u'''(0) = 0, \quad u(1) = 0, \text{ plus } \begin{cases} \text{either} & u'(1) = 0, & \text{(Clamped)} \\ \text{or} & u''(1) = 0. & \text{(Pinned)} \end{cases} \quad (11)$$

Since eigenfunctions are determined only up to a multiplicative constant, we add the normalization

$$u(0) = 1, \quad (12)$$

thereby selecting a unique solution of this problem.

Incidentally, we claim that (8) with either boundary conditions (11) is self-adjoint and in fact positive-definite. This may be proved with the usual integration-by-parts argument, with one subtlety: usually, boundary terms vanish because of the boundary conditions, but the term $(yu')'$ makes no contribution at $y = 0$ only because the coefficient y vanishes there. Thus, all eigenvalues of this problem will be positive real.

[†]In (3), the coefficient EI of the fourth derivative is appropriate for a solid rod but not for a *cable* or *string*—i.e., many small fibers twisted together. The bending resistance of any one fiber in a string is all but infinitesimal; the primary resistance comes from the friction of fibers sliding over one another. While it is difficult to calculate the bending resistance of such a collection of fibers, this resistance is *much* smaller than the Young's modulus times the area moment of the whole cable. Thus, appropriate values of ε for a string may be very small indeed. For accurate modeling of a string it might be necessary to include a friction term, say proportional to w_t , in (3). Such a term would not change the eigenfunctions found below, and its effect on the time dependence (9) is easily calculated.

The remainder of this paper is organized as follows. In Section 3 we propose a naive derivation of the limiting behavior of the eigenvalues and eigenfunctions of (8) as $\varepsilon \rightarrow 0$, and we present some numerical results. In Section 4 we find the boundary-layer scalings for the asymptotic analysis and solve the reduced equations in various regimes. In Sections 5 and 6 we derive matched asymptotic series for the eigenvalues and eigenfunctions with clamped and pinned boundary conditions, respectively. Finally, in Section 7 we discuss the applicability of small- ε asymptotics for later eigenvalues in the sequence of eigenvalues.

In addition to its research interest, this problem provides a relatively simple example of a matched asymptotic expansion that requires logarithmic terms. Thus for pedagogical reasons, we strive for careful, thorough explanations, and we have included several exercises for the readers to sharpen their understanding.

3 First Attempts

3.1 The Naive Solution: $\varepsilon = 0$

Setting ε equal to zero in (8) yields equation (10). Of course no solution of this equation can satisfy all the boundary conditions (11). It is natural to conjecture that at $y = 1$ the “most physical” solution will at least satisfy the lower-order boundary condition there

$$u(1) = 0. \quad (13)$$

Because (10) is singular at $y = 0$, it is unclear what boundary conditions, if any, ought to be imposed there.

In fact, (10) can be solved explicitly in terms of Bessel functions of order zero. The connection to Bessel functions may be motivated by expanding all terms in (10),

$$yu'' + u' + \lambda u = 0 \quad (14)$$

and observing that this equation, like Bessel’s equation, has exactly two singular points, a regular one at $y = 0$ and an irregular one at infinity. Moreover, the indicial equation of (14), obtained by seeking a series solution

$$u(y) = y^p \sum_{j=0}^{\infty} c_j y^j \quad (15)$$

where $c_0 \neq 0$, has a double root $p = 0$, again like Bessel’s equation of order zero. To make the reduction, consider the substitution $u(y) = v(Cy^p)$; a simple calculation shows that if one chooses $p = \frac{1}{2}$, $C = 2\sqrt{\lambda}$, then v satisfies Bessel’s equation $v'' + x^{-1}v' + v = 0$. Thus, the general solution of (10) is

$$u(y) = aJ_0\left(2\sqrt{\lambda y}\right) + bY_0\left(2\sqrt{\lambda y}\right) \quad (16)$$

for arbitrary constants a and b . As $y \rightarrow 0$, J_0 is smooth, but Y_0 blows up logarithmically (see Appendix A.2). Unless $a = b = 0$, neither of the boundary conditions (11) at $y = 0$ can be satisfied. Let us require that $b = 0$ so that at least u remains bounded (and (12) is meaningful). If the boundary condition (13) at $y = 1$ is to be satisfied, then λ must satisfy

$$J_0\left(2\sqrt{\lambda}\right) = 0. \quad (17)$$

This intuitive discussion gives eigenvalues $\lambda \approx 1.4458, 7.6178, 18.721, \dots$ with eigenfunctions $J_0(2\sqrt{\lambda y})$, for *either clamped or pinned* boundary conditions. The eigenfunctions are graphed

in Figure 1. Despite the many loose threads in the argument, exactly these eigenvalues and eigenfunctions will emerge as the leading-order term in our asymptotic solution, thus providing a far more satisfactory derivation. Moreover, the difference between clamped and pinned BC will emerge in the higher-order terms of the series. However, before tackling the asymptotics, we turn to numerics.

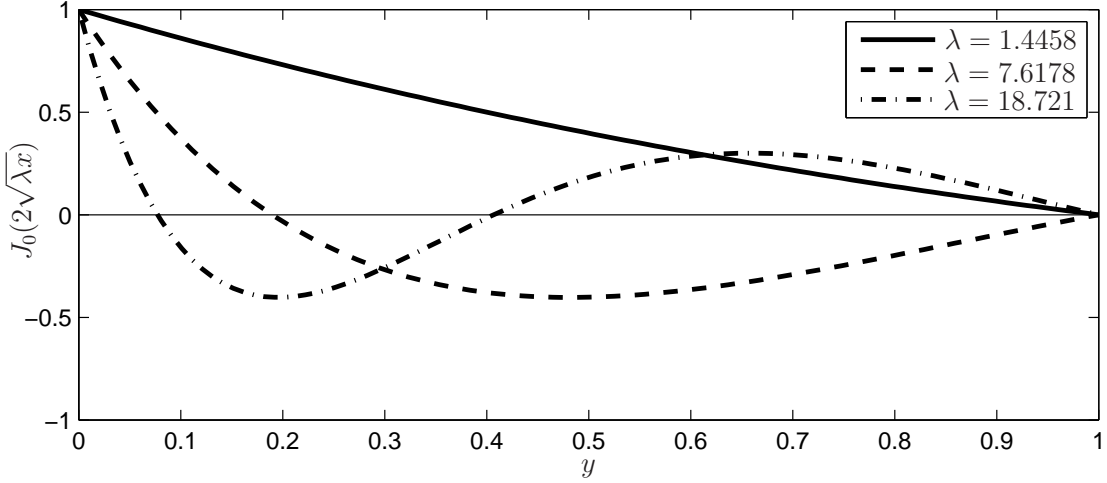


Figure 1: The naive solution of the reduced equation (10) using the three smallest roots of the Bessel function $J_0(2\sqrt{\lambda})$.

3.2 Numerical Solutions

In an exploratory numerical code, we approximated (8) by differences on a uniform grid. However, especially for small ε , we found this simple approach gave unreliable results, even when up to 50,000 grid points were used. For the data presented below, we used the boundary-value solver `bvp5c` in Matlab. The interval $0 \leq y \leq 1$ was divided into 3 distinct subintervals (corresponding to the 2 boundary layers and the bulk domain, introduced below), which were connected with internal boundary conditions. Results from the uniform-grid code were used as initial guesses for the internal iterative solver. Very small values of ε were approached by continuation.

Graphs of the computed eigenfunctions resembled the naive eigenfunctions in Figure 1, but graphs of their derivatives differed substantially. This issue is explored in some detail in Subsections 5.4 and 6.2(e).

Figure 2 shows the divergence of computed eigenvalues from the naive approximation; specifically, a log-log plot of $|\lambda^{(n)}(\varepsilon) - \lambda^{(n)}(0)| / \lambda^{(n)}(0)$ vs. ε , where $\lambda^{(n)}(0)$ is the n^{th} root of $J_0(2\sqrt{\lambda})$. The results support the naive analysis and also suggest that

$$\lambda^{(n)}(\varepsilon) = \lambda^{(n)}(0) + \mathcal{O}(\varepsilon^p) \quad (18)$$

where $p = 1/2$ or $p = 1$ for clamped or pinned boundary conditions, respectively. This suggestion is confirmed by the asymptotic series solution constructed below.

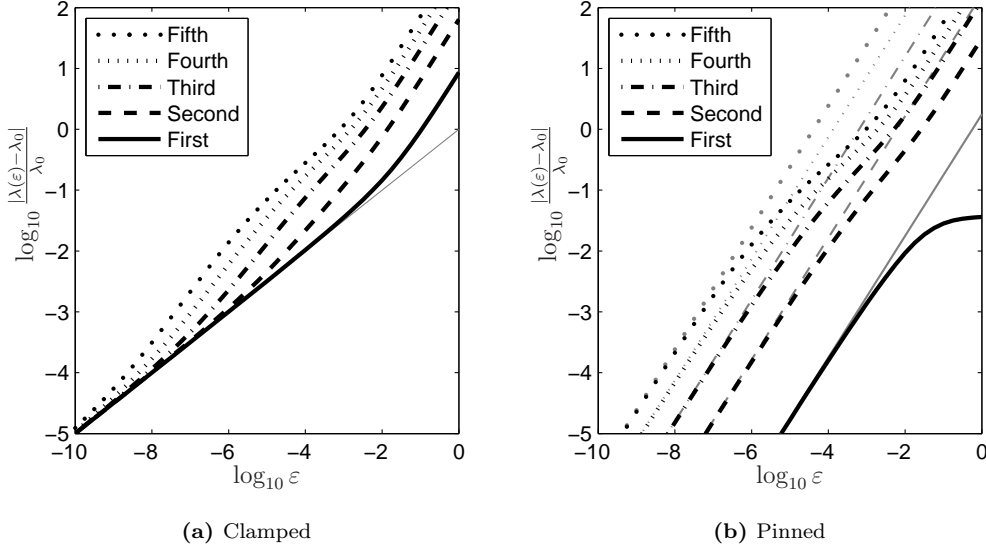


Figure 2: A log-log plot of the relative error $e(\varepsilon) = \left| \frac{\lambda^{(n)}(\varepsilon) - \lambda^{(n)}(0)}{\lambda^{(n)}(0)} \right|$ for the first five eigenvalues. According to our asymptotic analysis, in the clamped case, $e(\varepsilon) \sim (\lambda_{1/2}/\lambda_0)\varepsilon^{1/2}$ where $\lambda_{1/2}$ is given in Table 1; since $\lambda_{1/2} = \lambda_0$, all lines collapse onto $e(\varepsilon) = \varepsilon^{1/2}$ (which is drawn for reference). In the pinned case $e(\varepsilon) \sim (\lambda_1/\lambda_0)\varepsilon$ where λ_1 is given by (111). Reference lines $e(\varepsilon) = (\lambda_1/\lambda_0)\varepsilon$ are drawn in the figure.

4 Preparation for the Asymptotic Solutions

4.1 Boundary-Layer Scalings

In the bulk of the domain, far away from the boundaries, we assume that ε times the fourth derivative is small enough that we can ignore this term to lowest order, obtaining (10). As we have seen, solutions of this equation cannot satisfy all the boundary conditions. We expect therefore that there are two adjustment zones—boundary layers—connecting the solution in the bulk to the boundaries, in a way that satisfies the boundary conditions. Near the boundaries, we expect that the fourth derivative becomes large enough that, even when multiplied by ε , it cannot be ignored. As a first step we have to find the boundary-layer scalings that are appropriate near the boundaries.

To find the scaling near the $y = 0$ boundary, we assume that y is of order ε^p ; specifically that $u(y; \varepsilon)$ can be approximated by $U(X; \varepsilon)$ where $X = \varepsilon^p y$. To find p we differentiate according to Eq. (8) and find that

$$\underbrace{\varepsilon^{1+4p} U''''}_{a} - \underbrace{\varepsilon^p (XU')'}_{b} = \underbrace{\lambda U}_{c} \quad (19)$$

where prime indicates differentiation with respect to X . An appropriate scaling must balance Eq. (19); i.e., two terms must have the same order in ε and the third term must have order this high or higher. Thus, we look at three possible cases:

$a \approx b$ Thus $1 + 4p = p$ and so $p = -1/3$; in this case terms (a) and (b) are much larger (i.e., lower

order) than (c). Hence it is a good scaling of the boundary layer.

$b \approx c$ Thus $p = 0$ (*i.e.*, no scaling happens.) Here the two terms (b) and (c) are larger than (a), so this scaling is valid; indeed this is the bulk scaling.

$a \approx c$ Thus $1 + 4p = 0$ so $p = -1/4$ and so term (b) is much larger than (a) and (c), so this scaling does not balance.[‡]

The only boundary layer we find near $y = 0$ scales the inner variable X like $\varepsilon^{1/3}$, and the resulting equation is therefore

$$U'''' - (XU')' = \varepsilon^{1/3}\lambda U. \quad (20)$$

To find the scaling near at the $y = 1$ boundary, we write the solution $u(y; \varepsilon)$ as $V(Z; \varepsilon)$ where $Z = \varepsilon^q(1 - y)$. From Eq. (8) we obtain

$$\varepsilon^{1+4q}V'''' - \varepsilon^{2q}((1 - \varepsilon^{-q}Z)V')' = \lambda V. \quad (21)$$

Here there are also three possible cases.

Exercise 1. *In the same way as before, show that the only valid, interesting scaling for small ε at the $y = 1$ boundary has $q = -1/2$, and that the resulting equation for $V(Z; \varepsilon)$ is*

$$V'''' - \left((1 - \varepsilon^{1/2}Z)V'\right)' = \varepsilon\lambda V. \quad (22)$$

4.2 Reduced Equations

In the asymptotic series for the solution, terms of order $1/3$ and $1/2$ are forced by the boundary layers, and subsequent terms include all sums of integer multiples of $1/3$ and $1/2$. Thus, we look for a solution that has the following form:

- In the bulk we expand the solution $u(y; \varepsilon)$ as

$$u(y; \varepsilon) = u_0(y) + \varepsilon^{1/3}u_{1/3}(y) + \varepsilon^{1/2}u_{1/2}(y) + \varepsilon^{2/3}u_{2/3}(y) + \varepsilon^{5/6}u_{5/6}(y) + \varepsilon u_1(y) + \dots \quad (23)$$

- Near the $y = 0$ boundary, the solution is approximated by a similar asymptotic series for $U(X; \varepsilon)$, with $X = \varepsilon^{-1/3}y$.
- Similarly, for the solution near $y = 1$ with $V(Z; \varepsilon)$, where $Z = \varepsilon^{-1/2}(1 - y)$.
- The eigenvalue[§] λ is itself expanded as an asymptotic series:

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon^{1/3}\lambda_{1/3} + \varepsilon^{1/2}\lambda_{1/2} + \varepsilon^{2/3}\lambda_{2/3} + \varepsilon^{5/6}\lambda_{5/6} + \varepsilon\lambda_1 + \dots \quad (24)$$

Below, we will see that the bulk series needs to be augmented by a term of order $\varepsilon \log \varepsilon$, and logarithmic terms will be needed at higher order as well. However, for now we wait for this complication to arise naturally.

[‡]This scaling would be valid in the opposite asymptotic limit where one considers ε to be large.

[§]Of course, there is a sequence of eigenvalues with corresponding eigenfunctions. We drop the superscript in the $\lambda^{(n)}$ notation for sake of a cleaner presentation.

For the three domains, we substitute the asymptotic expansion of u , U , or V , and of λ into the appropriate differential equation and collect powers of ε . Doing this for an expansion up to ε^1 results in

$$\begin{aligned}
\varepsilon^0 : & \quad (yu'_0)' + \lambda_0 u_0 = 0 \\
\varepsilon^{1/3} : & \quad (yu'_{1/3})' + \lambda_0 u_{1/3} = -\lambda_{1/3} u_0 \\
\varepsilon^{1/2} : & \quad (yu'_{1/2})' + \lambda_0 u_{1/2} = -\lambda_{1/2} u_0 \\
\varepsilon^{2/3} : & \quad (yu'_{2/3})' + \lambda_0 u_{2/3} = -\lambda_{1/3} u_{1/3} - \lambda_{2/3} u_0 \\
\varepsilon^{5/6} : & \quad (yu'_{5/6})' + \lambda_0 u_{5/6} = -\lambda_{1/3} u_{1/2} - \lambda_{1/2} u_{1/3} - \lambda_{5/6} u_0 \\
\varepsilon^1 : & \quad (yu'_1)' + \lambda_0 u_1 = u_0''' - \lambda_{1/3} u_{2/3} - \lambda_{1/2} u_{1/2} - \lambda_{2/3} u_{1/3} - \lambda_1 u_0
\end{aligned} \tag{25}$$

for u ,

$$\begin{aligned}
\varepsilon^0 : & \quad U_0'''' - (XU_0')' = 0 \\
\varepsilon^{1/3} : & \quad U_{1/3}'''' - (XU_{1/3}')' = \lambda_0 U_0 \\
\varepsilon^{1/2} : & \quad U_{1/2}'''' - (XU_{1/2}')' = 0 \\
\varepsilon^{2/3} : & \quad U_{2/3}'''' - (XU_{2/3}')' = \lambda_0 U_{1/3} + \lambda_{1/3} U_0 \\
\varepsilon^{5/6} : & \quad U_{5/6}'''' - (XU_{5/6}')' = \lambda_0 U_{1/2} + \lambda_{1/2} U_0 \\
\varepsilon^1 : & \quad U_1'''' - (XU_1')' = \lambda_0 U_{2/3} + \lambda_{1/3} U_{1/3} + \lambda_{2/3} U_0
\end{aligned} \tag{26}$$

for U , and

$$\begin{aligned}
\varepsilon^0 : & \quad V_0'''' - V_0'' = 0 \\
\varepsilon^{1/3} : & \quad V_{1/3}'''' - V_{1/3}'' = 0 \\
\varepsilon^{1/2} : & \quad V_{1/2}'''' - V_{1/2}'' = -(ZV_0')' \\
\varepsilon^{2/3} : & \quad V_{2/3}'''' - V_{2/3}'' = 0 \\
\varepsilon^{5/6} : & \quad V_{5/6}'''' - V_{5/6}'' = -(ZV_{1/3}')' \\
\varepsilon^1 : & \quad V_1'''' - V_1'' = \lambda_0 V_0 - (ZV_{1/2}')'
\end{aligned} \tag{27}$$

for V .

4.3 Solutions of the Reduced Bulk Equations

In the bulk, the reduced equations (25) have the form $(yu')' + \lambda_0 u = g$, or expanding the derivative and dividing by y ,

$$u'' + \frac{1}{y}u' + \frac{\lambda_0}{y}u = \frac{g(y)}{y}. \tag{28}$$

We saw in Subsection 3.1 that in the homogeneous case, $g \equiv 0$, the general solution of (28) is given by (16). To shorten the notation and form a more convenient basis for the solution space, we define

$$\tilde{J}(y) = J_0(2\sqrt{\lambda_0 y}) \tag{29}$$

$$\tilde{Y}(y) = \pi Y_0(2\sqrt{\lambda_0 y}) - (\log \lambda_0 + 2\gamma)\tilde{J}(y) \tag{30}$$

where γ is Euler's constant. The modification of Y_0 simplifies the asymptotic behavior as $y \rightarrow 0$ (see Appendix A.2):

$$\tilde{Y}(y) = \log(y) + \mathcal{O}(y \log y). \tag{31}$$

Note that this notation hides λ_0 , an unknown constant that still needs to be determined.

In the inhomogeneous case we have

Lemma 1. *Given a continuous function $g(y)$, there exists exactly one solution of (28), say $w(y)$, that is continuous on the closed interval $[0, 1]$ and satisfies $w(0) = 0$.*

For this singular problem, just one boundary condition suffices to determine the solution uniquely.

Proof of Lemma 1: (Existence) Using variation of coefficients, we find that one solution of (28) is given by

$$w(y) = c_1(y)\tilde{J}(y) + c_2(y)\tilde{Y}(y), \quad (32)$$

where

$$c_1(y) = - \int_0^y \frac{1}{x\mathcal{W}(x)} \tilde{Y}(x)g(x)dx = - \int_0^y \tilde{Y}(x)g(x)dx \quad (33)$$

$$c_2(y) = \int_0^y \frac{1}{x\mathcal{W}(x)} \tilde{J}(x)g(x)dx = \int_0^y \tilde{J}(x)g(x)dx. \quad (34)$$

Here $\mathcal{W} = \tilde{J}\tilde{Y}' - \tilde{Y}\tilde{J}'$ denotes the Wronskian of $\tilde{J}(y)$ and $\tilde{Y}(y)$, and we have used the fact $\mathcal{W}(y) = 1/y$ (see Appendix A.2). Regarding the first term in (32), although the integrand defining c_1 in (33) diverges logarithmically at $x = 0$, the integral converges for all $y \in [0, 1]$ and vanishes if $y = 0$. Regarding the second term (34), although $\tilde{Y}(y)$ blows up logarithmically as $y \rightarrow 0$, the coefficient satisfies $c_2(y) = \mathcal{O}(y)$ so the product $c_2\tilde{Y}$ vanishes at $y = 0$. Thus, (32) solves (28) and satisfies the boundary conditions in Lemma 1.

(Uniqueness) The general solution of (28) has the form

$$u(y) = w(y) + a\tilde{J}(y) + b\tilde{Y}(y)$$

for arbitrary constants a and b . For this function to be continuous we need $b = 0$, and for it to vanish at the origin, we need $a = 0$. \square

Corollary 2. *If λ_0 satisfies $J_0(2\sqrt{\lambda_0}) = 0$ and $g(y) = \tilde{J}(y)$, then the solution to Eq. (28), $w(y)$, does not vanish at $y = 1$.*

Proof. Since $\tilde{J}(1) = 0$, we have from (32) that

$$w(1) = c_2(1)\tilde{Y}(1) = \tilde{Y}(1) \int_0^1 \tilde{J}^2(x) dx \neq 0. \quad (35)$$

Of course the integral is positive, and \tilde{Y} cannot vanish at $y = 1$ since \tilde{J} already vanishes there and the pair's Wronskian is non-zero. \square

4.4 Solution of the Reduced Boundary Layer Equations

Equations (27) for the boundary layer at $y = 1$, which have constant coefficients, do not require any special discussion. Therefore, we focus here on the boundary layer at $y = 0$.

The homogeneous versions of equations (26) have the form

$$U'''' - (XU')' = 0. \quad (36)$$

We need to find four linearly independent solutions of this equation. By inspection, $U^{(1)} \equiv 1$ is one such solution. Observe that, if we define $W = U'$, we may rewrite (36) as

$$\frac{d}{dX} [W'' - XW] = 0. \quad (37)$$

Now $W'' - XW = 0$ is Airy's differential equation, whose solution space is spanned by the Airy functions, Ai and Bi. (See Appendix A.1 for the definition and some elementary properties of these special functions.) Thus,

$$U^{(2)}(X) = \int_0^X \text{Ai}(x)dx, \quad U^{(3)}(X) = \int_0^X \text{Bi}(x)dx \quad (38)$$

provide two more linearly independent solutions. Since Bi grows super-exponentially as $X \rightarrow \infty$, the solution $U^{(3)}$ cannot be matched to any solution in the bulk. By contrast, Ai *decays* super-exponentially so the integral to infinity converges; in fact, by (120)

$$\lim_{X \rightarrow \infty} U^{(2)}(x) = 1/3. \quad (39)$$

Incidentally, for use in the boundary conditions (11, 12), we claim that

$$U^{(2)}(0) = 0, \quad U^{(2)''}(0) = \text{Ai}'(0) \neq 0, \quad U^{(2)'''}(0) = 0. \quad (40)$$

The first relation is trivial; the second may be derived by differentiating (38) twice; and third may be derived by differentiating (38) thrice and invoking Airy's differential equation.

To obtain a fourth independent solution, we satisfy (37) by requiring that $W'' - XW = -1$, or since $W = U'$,

$$U''' - XU' = -1. \quad (41)$$

Specifically we let $U^{(4)} = \Psi$ where Ψ satisfies the following

Lemma 3. *There is a unique solution $\Psi(X)$ of (41) such that*

$$\begin{aligned} (a) \quad & \Psi(0) = 0, \quad (b) \quad \Psi''(0) = 0 \quad \text{and} \\ (c) \quad & \Psi(X) = \log(X) + \mathcal{O}(1) \quad \text{as } X \rightarrow \infty. \end{aligned}$$

This lemma is proved in Appendix B, and Figure 6 shows the graph of Ψ , obtained numerically. Incidentally, the numerics indicate that as $X \rightarrow \infty$

$$\Psi(X) = \log(X) + \Psi_\infty + o(1) \quad \text{where} \quad \Psi_\infty \approx 1.3556, \quad (42)$$

while it follows from (41) that

$$\Psi'''(0) = -1. \quad (43)$$

Below we also solve inhomogeneous versions of equations (26).

5 Asymptotics for the Clamped Case

In this section, we calculate terms in the asymptotic series for the clamped case through the first non-trivial correction to the eigenvalue: i.e., order $1/2$. The results are summarized in Table 1.

α	λ_α	u_α	U_α	V_α
0	λ_0	$\tilde{J}(y)$	1	0
$\frac{1}{3}$	0	0	$-\lambda_0 X$	0
$\frac{1}{2}$	λ_0	$-\lambda_0 w(y)$	0	$\tilde{J}'(1) [1 - Z - e^{-Z}]$

Table 1: Coefficients of the asymptotic expansion for the clamped case. The leading-order eigenvalue λ_0 is a root of the equation $J_0(2\sqrt{\lambda_0}) = 0$. \tilde{J} and w are defined in (29) and (58), respectively.

5.1 Zeroth-Order Solution

(a) *Boundary-layer near $y = 0$:* As discussed in Section 4.4, the solution of the ε^0 -equation in (26) is a linear combination

$$U_0(X) = A + B \int_0^X \text{Ai}(x) dx + C\Psi(X), \quad (44)$$

the integral of Bi having been excluded as unsuitable for matching. To satisfy the boundary conditions (11, 12), we require

$$U_0(0) = 1, \quad U_0''(0) = 0, \quad U_0'''(0) = 0. \quad (45)$$

By substituting (44) into the boundary conditions and recalling the derivatives of $\int \text{Ai}$ and Ψ at $X = 0$, we deduce that $A = 1, B = C = 0$, that is,

$$U_0(X) \equiv 1. \quad (46)$$

(b) *Boundary-layer near $y = 1$:* The four-dimensional solution space of the ε^0 equation in (27) is spanned by $1, Z, e^{-Z}, e^Z$. Excluding e^Z as unsuitable for matching, we write $V_0(Z) = A + BZ + Ce^{-Z}$. The boundary conditions (11) for the clamped case,

$$V_0(0) = 0 \quad V_0'(0) = 0, \quad (47)$$

allow us to express two of the three arbitrary constants in terms of the third, yielding

$$V_0(Z) = A(1 - Z - e^{-Z}). \quad (48)$$

The undetermined coefficient A will be found by matching with the bulk solution.

(c) *Bulk:* We already know that the general solution of the ε^0 -equation (25) in the bulk is

$$u_0(y) = a\tilde{J}(y) + b\tilde{Y}(y). \quad (49)$$

The two constants will be determined in matching.

(d) *Matching:* To match the bulk solution with that of the boundary layers, we compare the “outer limit” of the inner solutions (the boundary-layer solutions) to the “inner limit” of the outer solution (the solution in the bulk). Thus, near $y = 0$ we need, as $\varepsilon \rightarrow 0$,

$$u_0(\varepsilon^{1/3}X) - U_0(X) = o(1) \quad (50)$$

for an appropriate range of X . Specifically, we require that there exist numbers p, q , with $0 \leq p < q \leq 1/3$, such that (50) holds for all X such that

$$\varepsilon^{-p} \ll X \ll \varepsilon^{-q}. \quad (51)$$

In this case we may take $p = 0, q = 1/3$; i.e., the maximal range. It follows from (49) that $u_0(y) = a + b \log y + \mathcal{O}(y \log y)$ for small y . Hence, since $\varepsilon^{1/3} X \ll 1$,

$$u_0(\varepsilon^{1/3} X) = a + b \log(\varepsilon^{1/3} X) + o(1). \quad (52)$$

Unless $b = 0$, the logarithm term in (52) is large and moreover depends on X . Therefore for u_0 to match onto $U_0(X) \equiv 1$, we need $a = 1$ and $b = 0$.

Near $y = 1$ we need

$$u_0(1 - \varepsilon^{1/2} Z) - V_0(Z) = o(1) \quad \text{for } Z \text{ in a range} \quad \varepsilon^{-p} \ll Z \ll \varepsilon^{-q} \quad (53)$$

where $0 \leq p < q \leq 1/2$. We take $p = 0, q = 1/2$. Given a, b as above, $u_0(1 - \varepsilon^{1/2} Z) = \tilde{J}(1) + o(1)$ while $V_0(Z) = A(1 - Z) + o(1)$. Thus matching the linear term in V_0 requires that $A = 0$, and then matching the constant terms requires that $\tilde{J}(1) = 0$. Finally, extracting the implicit λ_0 from the argument of \tilde{J} , we obtain the characterization of λ_0

$$J_0(2\sqrt{\lambda_0}) = 0. \quad (54)$$

This confirms the *ad hoc* solution we found in Subsection 3.1, and it verifies the first row of Table 1.

5.2 The $\varepsilon^{1/3}$ Correction

Not much happens at this order, but it serves as practice for later calculations.

(a) *Boundary-layer near $y = 0$:* At order $\varepsilon^{1/3}$, equation (26) is inhomogeneous with the right-hand-side $\lambda_0 U_0 \equiv \lambda_0$, which has the particular solution $U_p(X) = -\lambda_0 X$. $U_{1/3}$ must satisfy boundary conditions analogous to (45), except that now $U_{1/3}(0) = 0$ replaces the condition $U_0(0) = 1$. Since U_p already satisfies the boundary conditions, we have that

$$U_{1/3}(X) = -\lambda_0 X. \quad (55)$$

As we shall see below, this term merely matches the first derivative of the bulk solution u_0 at $y = 0$.

(b) *Boundary-layer near $y = 1$:* Arguing as in the ε^0 case in Subsection 5.1, we deduce that

$$V_{1/3}(Z) = A(1 - Z - e^{-Z}). \quad (56)$$

(c) *Bulk:* The $\varepsilon^{1/3}$ equation (25) has the inhomogeneous term $-\lambda_{1/3} u_0$. Thus, the general solution of this equation is

$$u_{1/3}(y) = -\lambda_{1/3} w(y) + a\tilde{J}(y) + b\tilde{Y}(y) \quad (57)$$

where we define w as the solution (see Lemma 1) of

$$(yw')' + \lambda_0 w = \tilde{J} \quad \text{such that} \quad w(0) = 0. \quad (58)$$

(d) *Matching:* Near $y = 0$ we need, as $\varepsilon \rightarrow 0$,

$$(u_0 + \varepsilon^{1/3}u_{1/3})(\varepsilon^{1/3}X) - (U_0 + \varepsilon^{1/3}U_{1/3})(X) = o(\varepsilon^{1/3}) \quad (59)$$

for X in a range $\varepsilon^{-p} \ll X \ll \varepsilon^{-q}$ where $0 \leq p < q \leq 1/3$. We take $p = 0, q = 1/6$ so that $(\varepsilon^{1/3}X)^2 = o(\varepsilon^{1/3})$. Then by Taylor expansion at $y = 0$,

$$u_0(\varepsilon^{1/3}X) = u_0(0) + \varepsilon^{1/3}u'_0(0)X + o(\varepsilon^{1/3}). \quad (60)$$

Therefore, obtaining $u'_0(0)$ from (126) in the Appendix, and recalling that $w(0) = 0$ in (57), we find[¶]

$$(u_0 + \varepsilon^{1/3}u_{1/3})(\varepsilon^{1/3}X) = 1 + \varepsilon^{1/3} \left(-\lambda_0 X + a + b \log(\varepsilon^{1/3}X) \right) + o(\varepsilon^{1/3}). \quad (61)$$

On the other hand,

$$(U_0 + \varepsilon^{1/3}U_{1/3})(X) = 1 - \varepsilon^{1/3}\lambda_0 X. \quad (62)$$

Thus (59) requires that $a = b = 0$.

Near $y = 1$ we need

$$(u_0 + \varepsilon^{1/3}u_{1/3})(1 - \varepsilon^{1/2}Z) - (V_0 + \varepsilon^{1/3}V_{1/3})(Z) = o(\varepsilon^{1/3}) \quad (63)$$

for Z in a range $\varepsilon^{-p} \ll Z \ll \varepsilon^{-q}$ where $0 \leq p < q \leq 1/2$. We take $p = 0, q = 1/6$ so that $\varepsilon^{1/2}Z = o(\varepsilon^{1/3})$. Now by a Taylor expansion near $y=1$

$$u_0(y) = -\tilde{J}'(1)(1 - y) + \mathcal{O}((1 - y)^2), \quad (64)$$

so by our choice of q

$$u_0(1 - \varepsilon^{1/2}Z) = \mathcal{O}(\varepsilon^{1/2}Z) = o(\varepsilon^{1/3}). \quad (65)$$

Therefore,

$$(u_0 + \varepsilon^{1/3}u_{1/3})(1 - \varepsilon^{1/2}Z) = \varepsilon^{1/3}u_{1/3}(1) + o(\varepsilon^{1/3}) = -\varepsilon^{1/3}\lambda_{1/3}w(1) + o(\varepsilon^{1/3}) \quad (66)$$

where we have recalled that $a = b = 0$ in (57). On the other hand, $V_0 \equiv 0$, so by (56), as $Z \rightarrow \infty$

$$(V_0 + \varepsilon^{1/3}V_{1/3})(Z) = A\varepsilon^{1/3}(1 - Z) + o(\varepsilon^{1/3}). \quad (67)$$

Thus matching requires that $A = 0$ and $\lambda_{1/3}w(1) = 0$. Recalling from Corollary 2 that $w(1) \neq 0$, we obtain $\lambda_{1/3} = 0$, and we have verified the second line of Table 1.

5.3 The $\varepsilon^{1/2}$ Correction

(a) *Boundary-layer near $y = 0$:* Equation (26), which at order $1/2$ is homogeneous, and the three homogeneous boundary conditions imply that $U_{1/2} \equiv 0$.

(b) *Boundary-layer near $y = 1$:* Since $V_0 \equiv 0$, equation (27) at order $1/2$ is homogeneous. By the same argument as for V_0 and $V_{1/3}$, this equation and the boundary conditions yield

$$V_{1/2}(Z) = A(1 - Z - e^{-Z}). \quad (68)$$

[¶]The term $\log(\varepsilon^{1/3}X)$ in this equation is a little disturbing. One expects the function that multiplies $\varepsilon^{1/3}$ to depend on X alone, not ε . This confusing behavior, a consequence of the singularity of (10) at $y = 0$, is not an issue here since the matching implies that $b = 0$. However, exactly this complication will force us to add a $\varepsilon \log \varepsilon$ term to the series in Section 6 below.

(c) *Bulk:* The $\varepsilon^{1/2}$ equation of (25) has the same family of solutions as in order $\varepsilon^{1/3}$:

$$u_{1/2}(y) = -\lambda_{1/2}w(y) + a\tilde{J}(y) + b\tilde{Y}(y) \quad (69)$$

where w is the solution of (58).

(d) *Matching:* Matching at $y = 0$ as above, we deduce that $a = b = 0$ in (69).

At $y = 1$ we need

$$(u_0 + \varepsilon^{1/3}u_{1/3} + \varepsilon^{1/2}u_{1/2})(1 - \varepsilon^{1/2}Z) - (V_0 + \varepsilon^{1/3}V_{1/3} + \varepsilon^{1/2}V_{1/2})(Z) = o(\varepsilon^{1/2}) \quad (70)$$

for Z in a range $\varepsilon^{-p} \ll Z \ll \varepsilon^{-q}$ where $0 \leq p < q \leq 1/2$. We choose $p = 0, q = 1/4$ so that $(\varepsilon^{1/2}Z)^2 = o(\varepsilon^{1/2})$. Now by (64) and (69),

$$(u_0 + \varepsilon^{1/3}u_{1/3} + \varepsilon^{1/2}u_{1/2})(1 - \varepsilon^{1/2}Z) = \varepsilon^{1/2} \left[-\tilde{J}'(1)Z - \lambda_{1/2}w(1) \right] + o(\varepsilon^{1/2}), \quad (71)$$

while letting $Z \rightarrow \infty$ in (68) we deduce

$$(V_0 + \varepsilon^{1/3}V_{1/3} + \varepsilon^{1/2}V_{1/2})(Z) = \varepsilon^{1/2}A(1 - Z) + o(\varepsilon^{1/2}). \quad (72)$$

To match the expressions in (70) we need $A = \tilde{J}'(1)$ and

$$\lambda_{1/2} = -\tilde{J}'(1)/w(1). \quad (73)$$

Remarkably, it follows from Lemma 4 that $\lambda_{1/2} = \lambda_0$. The verification of all entries in Table 1 is now complete.

As a check on our calculations, in Figure 5b below we present a log-log plot of the error in the two-term approximation $\lambda_0 + \lambda_{1/2}\varepsilon^{1/2}$ to the clamped eigenvalues. The resulting lines of slopes near 1 suggest that the next non-vanishing correction to the eigenvalue will happen at the $\mathcal{O}(\varepsilon)$ level. An exercise at the end of Section 6 includes verifying this point.

Lemma 4. *The function w satisfies*

$$w(1) = -\frac{\tilde{J}'(1)}{\lambda_0}$$

Proof. Recall formula (35) for $w(1)$. Manipulating [AS64, 11.3.34], we find that

$$\int_0^1 \tilde{J}^2(\tau) d\tau = \int_0^1 J_0^2(2\sqrt{\lambda_0}\tau) d\tau = J_0^2(2\sqrt{\lambda_0}) + J_1^2(2\sqrt{\lambda_0}). \quad (74)$$

Since λ_0 is a root of $J_0(2\sqrt{\lambda_0})$, we may drop the first term. Regarding the second we invoke [AS64, 9.1.28] to obtain

$$J_1^2(2\sqrt{\lambda_0}) = J_0'^2(2\sqrt{\lambda_0}) = \frac{1}{\lambda_0} \tilde{J}'^2(1). \quad (75)$$

Thus we obtain $w(1) = \tilde{Y}(1)\tilde{J}'(1)\frac{\tilde{J}'(1)}{\lambda_0}$. Since $\tilde{J}(1) = 0$,

$$\tilde{Y}(1)\tilde{J}'(1) = -\tilde{J}(1)\tilde{Y}'(1) + \tilde{Y}(1)\tilde{J}'(1) = -\mathcal{W}(1) = -1, \quad (76)$$

where we have taken the Wronskian from Appendix A.2. \square

5.4 Comparisons of Various Approximate Eigenfunctions

The outer and inner solutions in our asymptotic expansions of the eigenfunctions may be combined into a composite expansion that gives a uniformly accurate approximation in both regions. (See [Hin91, Section 5.1.8].) This may be formed by adding the inner and outer approximations, subtracting the common part in the matching region, and expressing the result as a function of the outer variable. This process is trivial at order 0: at $y = 1$ the inner approximation, and hence the common part, vanishes, while at $y = 0$, the inner approximation is nonzero, but it equals the common part, so the uniform approximation is simply the outer solution. It is similarly trivial at order $1/3$, so let us proceed to order $1/2$, where the behavior near $y = 1$ is interesting.

Inner and outer solutions are given in Table 1, and the common part of these expansions in the matching region near $y = 1$ is given by (71) or (72),

$$\varepsilon^{1/2} \tilde{J}'(1) [1 - Z]. \quad (77)$$

Thus, subtracting off the common part simply cancels the two polynomial terms in the inner solution, leaving only the exponential. Therefore, at order $1/2$, the composite approximation of the eigenfunctions is

$$\mathcal{U}_{1/2}(y) = \tilde{J}(y) - \varepsilon^{1/2} \lambda_0 \left[w(y) - w(1) e^{-(1-y)/\varepsilon^{1/2}} \right]. \quad (78)$$

Here we have used Lemma 4 to rewrite the coefficient of the exponential to make it obvious that $\mathcal{U}_{1/2}(0) = 0$. Although this boundary condition is satisfied exactly, the derivative boundary condition $u'(0) = 0$ is satisfied only to leading order; specifically $\mathcal{U}'_{1/2}(0) = \mathcal{O}(\varepsilon^{1/2})$. Such loss of accuracy in taking derivatives cannot be avoided.

Fig 3a shows the order-0 outer, the order-1/2 composite, and the order-1/2 inner approximations together with the numerical approximation to the first eigenfunction, while Figure 3b graphs the derivatives of the order-0 outer approximation and the order-1/2 composite approximation, all for $\varepsilon = 10^{-2}$. In anthropomorphic terms, $\mathcal{U}_{1/2}$ “attempts” to correct for the fact that the zeroth-order outer solution has nonzero derivative at $y = 1$. Thus, $\mathcal{U}_{1/2}$ is lowered in the interior of the interval so that it may approach $y = 1$ with nearly zero slope. Incidentally, the increase in the eigenvalue $\varepsilon^{1/2} \lambda_{1/2}$ is needed to drive $\mathcal{U}_{1/2}$ towards zero more rapidly in the interior.

In Fig 3a it may be seen that $\mathcal{U}_{1/2}$ differs noticeably from the numerically computed eigenfunction in the interior of the interval. This difference is corrected by the order-2/3 terms in the series, which are driven by the boundary layer at $y = 0$. (The order-2/3 corrections for clamped boundary conditions are the same as for the pinned case, which are calculated in Subsection 6.2.) In other words, for application to the flag problem mentioned in the introduction, it is unwise to neglect the order-2/3 corrections to the eigenfunction.

5.5 Lessons Learned

Having achieved our initial goal, let us pause to describe patterns in the calculations: Suppose all terms $u_\gamma, U_\gamma, V_\gamma, \lambda_\gamma$ of order $\mathcal{O}(\varepsilon^\gamma)$ for $\gamma < \alpha$ have been calculated, and consider what is needed to calculate the terms of order $\mathcal{O}(\varepsilon^\alpha)$.

(a) *Boundary-layer near $y = 0$:* The equation (26) for U_α ,

$$U'''' - (XU')' = \text{RHS}, \quad (79)$$

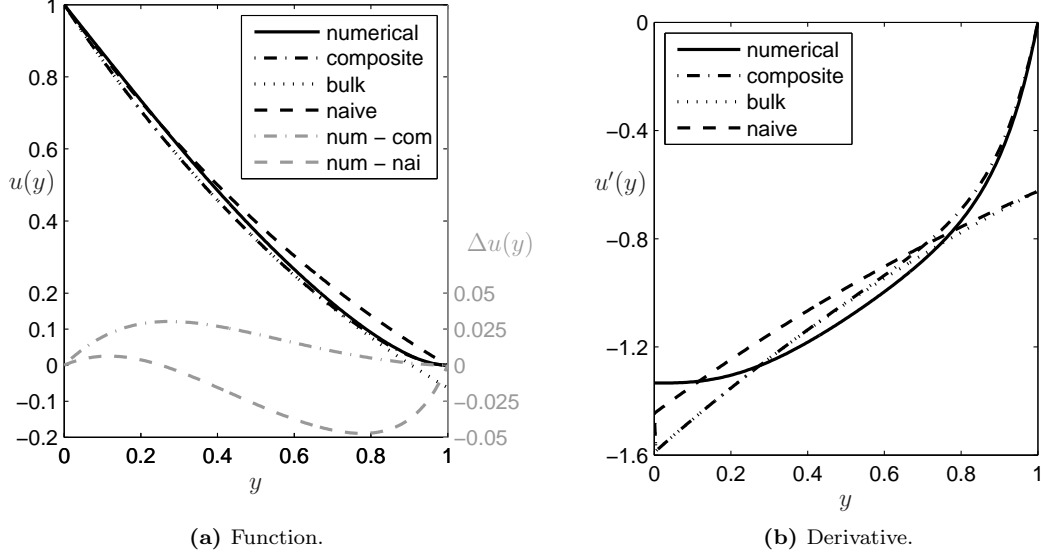


Figure 3: Comparison of numerical, naive, bulk, and composite approximations of order 1/2 to the first eigenfunction, for $\varepsilon = 10^{-2}$. Both function values and first derivative are shown; in Fig 3a, differences are overlaid for clarification.

has a four-dimensional solution space. However, because $\int \text{Bi}$ is excluded as unsuited for matching, only a three-dimensional space is available for forming U_α . This function must satisfy the three boundary conditions at $X = 0$ derived from (11,12); one expects these boundary conditions to determine U_α uniquely.

The particular and homogeneous solutions of (79) play different roles. The particular solution, which has polynomial growth as $X \rightarrow \infty$, matches onto derivatives of lower-order terms in the bulk series. (For example, the particular solution $-\lambda_0 X$ in $U_{1/3}$ matched onto $u_0(\varepsilon^{1/3} X)$.) The homogeneous solution, a linear combination $A + B \int \text{Ai} + C\Psi(X)$, matches onto u_α , the bulk solution of the same order. Note that as $X \rightarrow \infty$

$$A + B \int_0^X \text{Ai}(x) dx + C\Psi(X) = C \log X + (A + B/3 + C\Psi_\infty) + o(1). \quad (80)$$

The three-dimensional parameter space is constrained in two directions by matching onto u_α . The remaining degree of freedom provides the additional flexibility needed to satisfy all the boundary conditions.

(b) *Boundary-layer near $y = 1$:* The equation (27) for V_α ,

$$V'''' - V'' = \text{RHS}, \quad (81)$$

has a three-dimensional space of solutions appropriate for matching. Solutions of the homogeneous equation are spanned by $1, Z, e^{-Z}$. The constant function matches onto u_α , the bulk solution of the same order; Z matches onto $u_{\alpha-1/2}(1 - \varepsilon^{1/2} Z)$; and e^{-Z} provides the flexibility needed to

satisfy all boundary conditions without contributing to the matching. If $\alpha \geq 1$, then a particular solution of (81) will also be needed to match onto derivatives of $u_{\alpha-1}, u_{\alpha-3/2}, \dots$.

The general solution of (81) has the form

$$V_\alpha(Z) = A + BZ + Ce^{-Z} + V_p(Z). \quad (82)$$

The constants may be determined by satisfying three equations: two come from the boundary conditions at $Z = 0$ and the third arises from matching terms proportional to Z as $Z \rightarrow \infty$ with the derivative of $u_{\alpha-1/2}$ at $y = 1$. Once V_α is determined, matching to u_α yields an effective boundary condition for u_α at $y = 1$.

It is interesting to compare matching at the two end points. In both cases, matching constrains the inner solution in two directions. At $y = 0$, both directions relate to u_α , the outer solution at the order being calculated. In contrast, at $y = 1$, one direction relates to u_α and the other to $u_{\alpha-1/2}$.

(c) *Bulk:* In the equation (25) for u_α , let us split off the term on the right proportional to the yet-to-be-calculated coefficient λ_α :

$$(yu')' + \lambda_0 u = -\lambda_\alpha u_0 + R.$$

The general solution of this equation has the form

$$u_\alpha = -\lambda_\alpha w + u_p + a\tilde{J} + b\tilde{Y}, \quad (83)$$

where w is the solution of (58) and a, b are arbitrary. If it weren't for the complications arising from the logarithmic behavior of \tilde{Y} near $y = 0$, the remaining steps would be extremely simple: Matching u_α to U_α at $y = 0$ provides two equations, which we may use to determine a and b in (83). Then the effective boundary condition from matching at $y = 1$ provides a linear equation for λ_α , completing the calculation to order $\mathcal{O}(\varepsilon^\alpha)$. These simple ideas suffice until we encounter order $\alpha = 1$; even then, the complications are rather mild.

(d) *Matching:* To match at $y = 0$ we need

$$\sum_{\gamma \leq \alpha} \varepsilon^\gamma \left[u_\gamma(\varepsilon^{1/3} X) - U_\gamma(X) \right] = o(\varepsilon^\alpha) \quad (84)$$

for an appropriate range of X . Since all terms of order less than α have already been matched, we may focus only on terms of order exactly α . In the inner series, only U_α contributes a term of order exactly α :

$$\sum_{\gamma \leq \alpha} \varepsilon^\gamma U_\gamma(X) = \dots + \varepsilon^\alpha U_\alpha(X) + o(\varepsilon^\alpha)$$

where \dots indicates terms of order less than α , which are not relevant for calculating the $\mathcal{O}(\varepsilon^\alpha)$ -solution. By contrast, in the outer series, in addition to u_α , derivatives of lower-order terms in the series also contribute terms of exactly this order. Of course these lower-order terms in the outer solution have already been determined; typically they are matched by the particular-solution part of U_α , the arbitrary constants in the homogeneous solution playing no role. The “bleeding” of lower-order terms into the order- α matching is also responsible for the fact that the matching interval shrinks as α increases.

Similar considerations apply to matching at $y = 1$.

We have everywhere performed matching in terms of the inner variable. It is possible to use the outer variable instead, but in our opinion the calculations are less clear: Specifically, the terms needing to be matched at order α are precisely those that, when expressed in terms of the inner variable, are proportional to ε^α .

6 Asymptotics for the Pinned Case

6.1 The Low-Order Solution

We now consider pinned boundary conditions at $y = 1$ with the same conditions at $y = 0$:

$$u''(0) = 0, \quad u'''(0) = 0, \quad u(1) = 0, \quad u''(1) = 0, \quad (85)$$

plus the normalization $u(0) = 1$. We calculate the terms in the asymptotic series through the first nontrivial correction to the eigenvalue—in this case order one. The results are summarized in Table 2.

α	λ_α	u_α	U_α	V_α
0	λ_0	$\tilde{J}(y)$	1	0
1/3	0	0	$-\lambda_0 X$	0
1/2	0	0	0	$-\tilde{J}'(1)Z$
2/3	0	$-\frac{\lambda_0^2}{6\text{Ai}'(0)}\tilde{J}(y)$	$\frac{\lambda_0^2}{4}X^2 - \frac{\lambda_0^2}{2\text{Ai}'(0)}\int_0^X \text{Ai}(x) dx$	0
5/6	0	0	0	0
1	(111)	(112)	(94)	(97)

Table 2: The coefficients of the asymptotic expansion for the pinned case. The last row contains the equation numbers where the terms, too long and cumbersome for the table, can be found.

Exercise 2. *Derive the first 3 rows of Table 2.*

These first few orders are very similar to the clamped case.

6.2 The $\varepsilon^{2/3}$ Correction Term

(a) *Boundary-layer near $y = 0$:* Using the values of $U_{1/3}$ and $\lambda_{1/3}$ from Table 2, the reduced equation (26) for $U_{2/3}$ has the inhomogeneous term $-\lambda_0^2 X$, with particular solution $U_p = \frac{\lambda_0^2}{4} X^2$. Therefore, the relevant family of solutions is

$$U_{2/3}(X) = \frac{\lambda_0^2}{4} X^2 + A + B \int_0^X \text{Ai}(\tau) d\tau + C \cdot \Psi(X). \quad (86)$$

The boundary conditions $U_{2/3}(0) = U_{2/3}''(0) = U_{2/3}'''(0) = 0$ determine that $A = C = 0$ and $B = -\frac{\lambda_0^2}{2\text{Ai}'(0)}$. So we have verified the table entry for $U_{2/3}$.

(b) *Boundary-layer near $y = 1$:* Equation (27) together with the two boundary conditions at $y = 1$ imply that $V_{2/3}$ has the form

$$V_{2/3}(Z) = A(1 - Z - e^{-Z}). \quad (87)$$

(c) *Bulk:* Because $\lambda_{1/3}$ vanishes, the ODE (25) for $u_{2/3}$ has only the inhomogeneous term $-\lambda_{2/3}u_0$. The general solution of this equation is

$$u_{2/3}(y) = -\lambda_{2/3}w(y) + a\tilde{J}(y) + b\tilde{Y}(y),$$

where $w(y)$ is the function that solves (58).

(d) *Matching:* At $y = 0$ we need equation (84), with $\alpha = 2/3$, to hold for X in the range $\varepsilon^{-p} \ll X \ll \varepsilon^{-q}$ where $0 \leq p < q \leq 1/3$. We take $p = 0, q = 1/9$ so that $(\varepsilon^{1/3}X)^3 = o(\varepsilon^{2/3})$, and we focus only on terms of order exactly $2/3$, using ellipsis to represent terms of lower order. From (c) and using the fact that $w(0) = 0$,

$$\sum_{\gamma \leq 2/3} \varepsilon^\gamma u_\gamma(\varepsilon^{1/3}X) = \dots + \varepsilon^{2/3} \left[\frac{u_0''(0)}{2} X^2 + a + b \log(\varepsilon^{1/3}X) \right] + o(\varepsilon^{2/3}). \quad (88)$$

On the other hand, reading $U_{2/3}$ from Table 2 and using (120) from the Appendix A.1,

$$\sum_{\gamma \leq 2/3} \varepsilon^\gamma U_\gamma(X) = \dots + \varepsilon^{2/3} \left[\frac{\lambda_0^2}{4} X^2 - \frac{\lambda_0^2}{6\text{Ai}'(0)} \right] + o(\varepsilon). \quad (89)$$

We see from the Taylor series (126) that the quadratic terms in (88) and (89) agree, as expected. Matching the new terms at order $2/3$ gives

$$a = -\frac{\lambda_0^2}{6\text{Ai}'(0)}, \quad b = 0. \quad (90)$$

At $y = 1$ we match for Z in the range $\varepsilon^{-p} \ll Z \ll \varepsilon^{-q}$ where we take $p = 0, q = 1/6$ so that $(\varepsilon^{1/2}Z)^2 = o(\varepsilon^{2/3})$. Again we focus only on terms of order exactly $2/3$. Now

$$\sum_{\gamma \leq 2/3} \varepsilon^\gamma u_\gamma(1 - \varepsilon^{1/2}Z) = \dots + \varepsilon^{2/3} u_{2/3}(1) + o(\varepsilon^{2/3});$$

no derivative of the bulk solution bleeds into the $\varepsilon^{2/3}$ term since there is no term of order $1/6$. At the same time, by (87)

$$\sum_{\gamma \leq 2/3} \varepsilon^\gamma U_\gamma(Z) = \dots + A\varepsilon^{2/3}(1 - Z) + o(\varepsilon^{2/3}).$$

Thus matching requires $A = 0$ and $u_{2/3}(1) = 0$; the latter may be written out as

$$-\lambda_{2/3}w(1) + a\tilde{J}(1) = 0$$

where a is given by (90). Of course $\tilde{J}(1) = 0$ and $w(1) \neq 0$, thus it follows that $\lambda_{2/3} = 0$ and $u_{2/3}$ is as given in Table 2.

(e) *The composite approximation:* As discussed in Subsection 5.4, the composite approximation is given by the sum of the outer and inner approximations minus the matching terms. Taking the outer and inner solutions from Table 2 and the matching terms near $y = 0$ from (89), we find

$$\mathcal{U}_{2/3}(y) = \tilde{J}(y) - \varepsilon^{2/3} \frac{\lambda_0^2}{6\text{Ai}'(0)} \left(\tilde{J}(y) - 3 \int_{\varepsilon^{-1/3}y}^{\infty} \text{Ai}(x) dx \right). \quad (91)$$

Here the integral term results from a convenient cancellation: in the matching layer, as $X \rightarrow \infty$, the integral over $(0, X)$ tends to the integral over $(0, \infty)$, so the inner solution minus the matching terms simplifies to the integral over (X, ∞) .

Figure 4a compares the numerical solution and two other approximations to the third eigenfunction: the naive approximation and the order-2/3 composite approximation. The graphs of all three functions resemble that of Figure 1, and to visual accuracy they coincide; therefore we have plotted only differences. Figure 4b graphs the second derivatives of these three approximate eigenfunctions. In all plots $\varepsilon = 10^{-5}$; because we are examining the *third* eigenfunction, we need a smaller ε than in Subsection 5.4 in order for the asymptotics to be meaningful—see Section 7. The order-2/3 composite approximation captures most of the boundary-layer behavior of the eigenfunction. In particular, observing in the Taylor series (126) that $\tilde{J}''(0) = \frac{\lambda_0^2}{2}$, we compute that

$$\mathcal{U}_{2/3}''(0) = -\frac{\lambda_0^4}{12\text{Ai}'(0)} \varepsilon^{2/3};$$

thus, the second derivative vanishes to leading order.

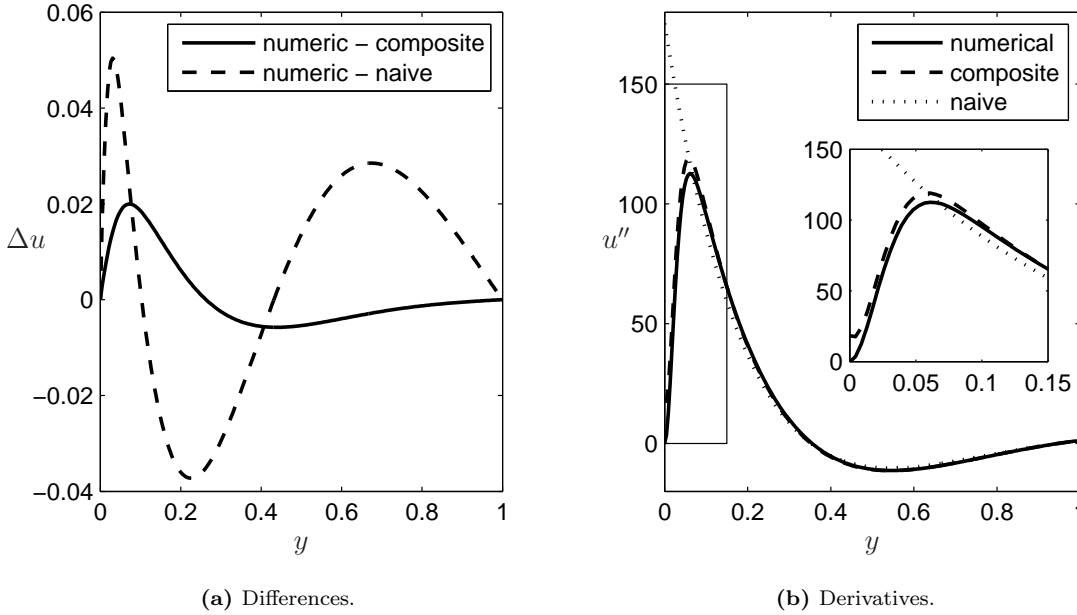


Figure 4: Comparison of (a) function values and (b) second derivatives of order-2/3 approximations to the third eigenvalue, for $\varepsilon = 10^{-5}$. Since the function values are too close to discern visually, various differences between functions are shown instead.

6.3 The $\varepsilon^{5/6}$ Correction

Exercise 3. Show that all the terms in this order vanish, as shown in Table 2.

6.4 The ε^1 Correction

This order of the calculation is pedagogically interesting because a logarithmic term enters the series.

(a) The $y = 0$ boundary: Equation (26) for U_1 is

$$U_1'''' - (XU_1')' = \frac{\lambda_0^3}{4}X^2 - \frac{\lambda_0^3}{2\text{Ai}'(0)} \int_0^X \text{Ai}(x)dx. \quad (92)$$

This looks quite bad due to the integral of Ai in the RHS, but in Exercise 4 you are guided through a proof that

$$G(X) = -\frac{\lambda_0^3}{36}X^3 + \frac{\lambda_0^3}{2\text{Ai}'(0)} \int_0^X (X-x)\text{Ai}(x)dx \quad (93)$$

is a particular solution to (92), and in Exercise 5 you are asked to apply boundary conditions to conclude that

$$U_1(X) = G(X) - \frac{\lambda_0^3}{2} \frac{\text{Ai}(0)}{(\text{Ai}'(0))^2} \int_0^X \text{Ai}(x)dx + \frac{\lambda_0^3}{3}\Psi(X). \quad (94)$$

Exercise 4. Find a particular solution to (92):

1. Show that the derivative of $\int_0^X (X-x)\text{Ai}(x)dx$ equals $\int_0^X \text{Ai}(x)dx$.
2. Use this when applying the LHS of ODE (92) to $\int_0^X (X-x)\text{Ai}(x)dx$ and obtain $-\int_0^X \text{Ai}(x)dx$.
3. Conclude that $G(X)$ in (93) is a particular solution of (92) that grows only polynomially at large values of X .

Exercise 5. Find the $y = 0$ boundary layer solution U_1 :

1. Using Airy's differential equation, show that

$$G(0) = 0, \quad G''(0) = \frac{\lambda_0^3}{2} \frac{\text{Ai}(0)}{\text{Ai}'(0)}, \quad G'''(0) = \frac{\lambda_0^3}{3}. \quad (95)$$

2. The family of possible solutions for U_1 is

$$U_1(X) = G(X) + A + B \int_0^X \text{Ai}(x)dx + C\Psi(X). \quad (96)$$

Using what you just found about G , show that (94) satisfies the boundary conditions $U_1(0) = U_1''(0) = U_1'''(0) = 0$.

(b) *The $y = 1$ boundary:* The boundary layer equation is $V_1'''' - V_1'' = \tilde{J}'(1)$ whose family of possible solutions is

$$V_1 = -\frac{\tilde{J}'(1)}{2}Z^2 + A + BZ + Ce^{-Z}.$$

The boundary conditions $V(0) = V''(0) = 0$ determine A and C so that

$$V_1 = -\tilde{J}'(1) \left[\frac{1}{2}Z^2 + 1 - e^{-Z} \right] + BZ, \quad (97)$$

and by matching to the bulk solution (below), it is found that $B = 0$.

(c) *Bulk:* The bulk equation for u_1 is familiar, albeit with a new term on the RHS:

$$(yu_1')' + \lambda_0 u_1 = -\lambda_1 u_0 + u_0'''. \quad (98)$$

Its general solution is

$$u_1(y) = -\lambda_1 w(y) + v(y) + a\tilde{J}(y) + b\tilde{Y}(y), \quad (99)$$

where $w(y)$ is defined by (58) and $v(y)$ is the solution to

$$(yv')' + \lambda_0 v = \tilde{J}'''(y), \quad v(0) = 0. \quad (100)$$

(Existence and uniqueness of $v(y)$ provided by Lemma 1.)

(d) *Matching:* Near $y = 0$ we need (84) to hold for a range of X such that $\varepsilon^{-p} \ll X \ll \varepsilon^{-q}$ where $0 \leq p < q \leq \frac{1}{3}$. We take $p = 0$, $q = 1/12$ so that $(\varepsilon^{1/3}X)^4 = o(\varepsilon^1)$. Using an ellipsis for terms of order less than 1, we have for u

$$\sum_{\gamma \leq 1} \varepsilon^\gamma u_\gamma(\varepsilon^{1/3}X) = \dots + \varepsilon \left[\frac{\tilde{J}'''(0)}{6}X^3 - \frac{\lambda_0^2 \tilde{J}'(0)}{6\text{Ai}'(0)}X + a + b \left(\frac{1}{3} \log \varepsilon + \log X \right) \right] + o(\varepsilon^1); \quad (101)$$

the cubic and linear terms come from derivatives of u_0 and $u_{2/3}$ respectively. At large X , the behavior of U_1 is

$$\sum_{\gamma \leq 1} \varepsilon^\gamma U_\gamma(X) = \dots + \varepsilon \lambda_0^3 \left[-\frac{1}{36}X^3 + \frac{1}{6\text{Ai}'(0)}X + C_\infty + \frac{1}{3} \log X \right] + o(\varepsilon^1) \quad (102)$$

where

$$C_\infty = \frac{1}{2} - \frac{\text{Ai}(0)}{6(\text{Ai}'(0))^2} + \frac{1}{3}\Psi_\infty \approx 0.06855. \quad (103)$$

Here the $1/2$ in C_∞ comes from a term in $G(X)$, using the relation

$$\int_0^\infty x \text{Ai}(x) dx = -\text{Ai}'(0)$$

obtained from the Airy equation. It is readily seen from (126) that the cubic and linear terms of (101) and (102) match, and the logarithmic terms will match if $b = \lambda_0^3/3$. However, it is not possible to match (101) and (102) with a coefficient a that is independent of ε . To fix this problem we propose to augment the u -series with a logarithmic term: i.e., to replace $\varepsilon u_1(y)$ by

$$\varepsilon [(\log \varepsilon) \hat{u}_1(y) + u_1(y)]. \quad (104)$$

In Exercise 6 below we ask the reader to show that:

- No such log terms are possible in the asymptotic series for either boundary layer or for λ .
- The equation for \hat{u}_1 is just the homogeneous version of (25) so that for some coefficients \hat{a}, \hat{b}

$$\hat{u}_1(y) = \hat{a}\tilde{J}(y) + \hat{b}\tilde{Y}(y). \quad (105)$$

Therefore in matching, the RHS of (101) should be replaced by

$$\dots + \varepsilon \left[\frac{\tilde{J}'''(0)}{6} X^3 - \frac{\lambda_0^2 \tilde{J}'(0)}{6A\tilde{I}'(0)} X + (a + \hat{a} \log \varepsilon) + (b + \hat{b} \log \varepsilon) \left(\frac{1}{3} \log \varepsilon + \log X \right) \right] + o(\varepsilon^1). \quad (106)$$

Matching between (102) and (106) is now possible if and only if

$$a = C_\infty, \quad b = \lambda_0^3/3, \quad \hat{a} = -b/3 = -\lambda_0^3/9, \quad \hat{b} = 0. \quad (107)$$

Near the boundary at $y = 1$ we match for Z in the range $1 \ll Z \ll \varepsilon^{-1/6}$ so that both $(\varepsilon^{1/2}Z)^3 = o(\varepsilon^1)$ (to allow neglect of u_0''') and $\varepsilon^{2/3} \cdot \varepsilon^{1/2}Z = o(\varepsilon^1)$ (to allow neglect of $u_{2/3}'$). As $Z \rightarrow \infty$, the u -series has the asymptotic behavior

$$\dots + \varepsilon \left[u_0''(1) \frac{Z^2}{2} + u_1(1) \right] + o(\varepsilon); \quad (108)$$

mercifully the contribution of \hat{u}_1 , which is proportional to $\tilde{J}(1)$, vanishes. From (97) the V -series has asymptotic behavior

$$\dots + \varepsilon \left[-\tilde{J}'(1) \left(\frac{Z^2}{2} + 1 \right) + BZ \right] + o(\varepsilon). \quad (109)$$

According to Exercise 7 below, the quadratic terms match; for the linear terms to match we need $B = 0$; and for the constant terms to match we need

$$u_1(1) = -\tilde{J}'(1). \quad (110)$$

Thus manipulating (99), we deduce that

$$\lambda_1 = \frac{v(1) + \frac{\lambda_0^3}{3} \tilde{Y}(1) + \tilde{J}'(1)}{w(1)} = \lambda_0 \frac{\int_0^1 \tilde{J}''''(\tau) \tilde{J}(\tau) d\tau}{[\tilde{J}'(1)]^2} + \frac{1}{3} \frac{\lambda_0^4}{[\tilde{J}'(1)]^2} - \lambda_0, \quad (111)$$

the latter equality following from the proof of Lemma 1, from Lemma 4, and (76). We may regard the three terms in (111) as contributions from the fourth derivative in (8), the boundary conditions at $y = 0$, and the boundary conditions at $y = 1$, respectively. (Numerically, we find that $\lambda_1 \approx 4.4280, 1887.2$ and 44403 for the first three eigenvalues.) Finally, we have

$$(\log \varepsilon) \hat{u}_1(y) + u_1(y) = -\lambda_1 w(y) + v(y) + (C_\infty - \frac{\lambda_0^3}{9} \log \varepsilon) \tilde{J}(y) + \frac{\lambda_0^3}{3} \tilde{Y}(y) \quad (112)$$

where C_∞ is given by (103).

As a check on our calculations, in Figure 5a we present a log-log plot of the error in the two-term approximation $\lambda_0 + \lambda_1 \varepsilon$ to the eigenvalue. The remaining error in the approximation seems to have a slope of $4/3$.

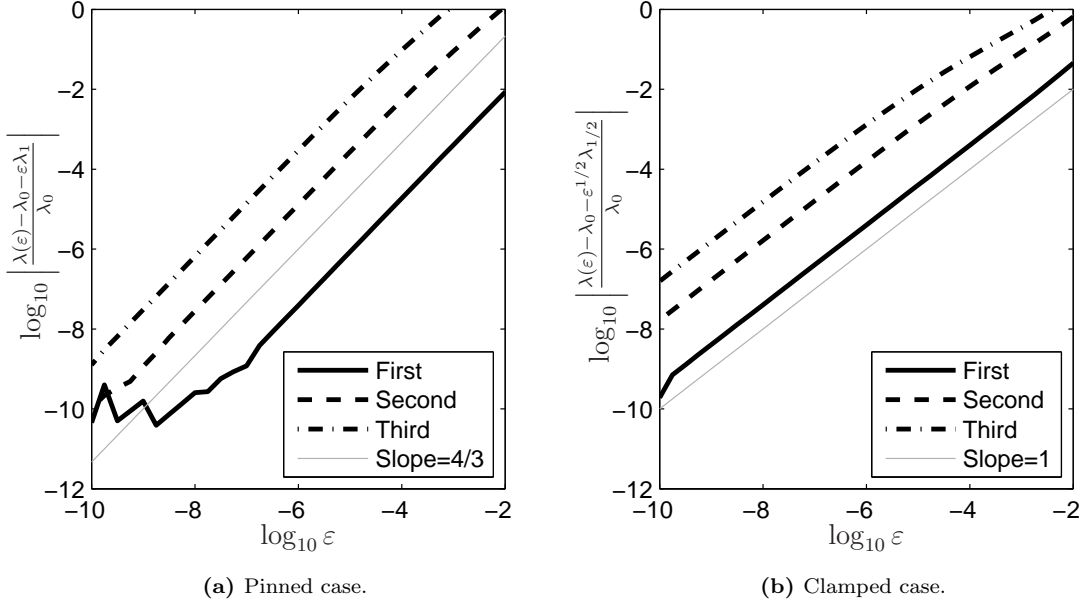


Figure 5: A log-log plot of the error of the two-term approximation to the eigenvalues, together with a line of appropriate slope for visual reference.

Exercise 6. Suppose that all of the series—for u , U , V , and λ —contain terms of order $\varepsilon \log \varepsilon$, with the relevant functions identified by a “hat.”

1. Deduce that $\hat{U}_1(X) \equiv 0$ because it satisfies a homogeneous differential equation of the type (26) with homogeneous boundary conditions.
2. Deduce from the (homogeneous) equation for \hat{V}_1 , of the type (27), and the (homogeneous) boundary conditions that $\hat{V}_1(Z) = BZ$ for some constant B . Show by matching with the lower-order parts of the inner solution that $B = 0$.
3. Observe that $\hat{u}_1(y)$ will satisfy an ODE of the type (25)

$$(y\hat{u}'_1(y))' + \lambda_0\hat{u}_1(y) = \hat{\lambda}_1 u_0.$$

Deduce from this equation plus matching that $\hat{\lambda}_1 = 0$ and \hat{u}_1 has the form (105).

Exercise 7. Using the fact that $\tilde{J}(1) = 0$, deduce from ODE (25) for u_0 that

$$\tilde{J}'(1) = -\tilde{J}''(1).$$

We invite the ambitious reader to calculate the additional terms in asymptotic series for the *clamped* case through $\mathcal{O}(\varepsilon)$: The order-2/3 terms are identical and the order-5/6 terms have one minor difference, while there are some significant differences at order-1. For example, the correction for the eigenvalue is given by

$$\lambda_1 = \frac{v(1) + \lambda_0^2 r(1) + \frac{\lambda_0^3}{3} \tilde{Y}(1) - \frac{3}{4} \tilde{J}'(1) + \lambda_0 w'(1)}{w(1)}, \quad (113)$$

where r is defined by

$$(yr')' + \lambda_0 r = w, \quad r(0) = 0.$$

Numerically, $\lambda_1 \approx 6.4581, 1900$ and 44435 for the first three eigenvalues.

7 Closing Remarks

As mentioned in the introduction, the aim of this paper was to understand the effect on the eigenfunctions for (6) of a small, but nonzero, bending resistance. We have found that, for small epsilon, the order-2/3 composite approximation (91) provides a reasonable correction to the naive eigenfunctions. However, the phrase “small epsilon” needs clarification.

Our notation so far hides the fact that problem (8) has an infinite sequence of eigenvalues $\lambda^{(n)}(\varepsilon)$ that tend to infinity as $n \rightarrow \infty$. For any fixed n , $\lambda^{(n)}(\varepsilon)$ has an asymptotic series in ε with leading order $\lambda_0^{(n)}$ characterized by (17). We ask, for a given n , how small must ε be to get into the asymptotic range. A minimal requirement is that the order-2/3 correction to the eigenfunction be small compared to unity: i.e.,

$$\varepsilon^{2/3} \lambda_0^2 \ll 1. \quad (114)$$

To make this more quantitative, we invoke the asymptotic approximation (122) of the Bessel function to deduce that, as $n \rightarrow \infty$

$$\lambda_0^{(n)} \sim (n - \tfrac{1}{4})^2 \frac{\pi^2}{4}. \quad (115)$$

Thus, (114) requires that

$$\varepsilon \ll n^{-6}. \quad (116)$$

In other words, the meaning of “small epsilon” depends heavily on n .

In fact, for some purposes, even (116) is not sufficiently restrictive. Let us define the asymptotic region by requiring that, in Figure 2, say in the clamped case, the graph of $\lambda^{(n)}(\varepsilon) - \lambda_0^{(n)}$ has converged to a line of slope 1/2; in symbols,

$$|\lambda^{(n)}(\varepsilon) - \lambda_0^{(n)} - \varepsilon^{1/2} \lambda_{1/2}^{(n)}| \ll \varepsilon^{1/2} \lambda_{1/2}^{(n)}. \quad (117)$$

We may estimate the LHS of (117) by the next term in the asymptotic series, $\varepsilon \lambda_1^{(n)}$, and it may be shown by estimating the terms of (113) that $\lambda_1^{(n)} = \mathcal{O}(n^7)$. Since $\lambda_{1/2}^{(n)} = \lambda_0^{(n)}$ and by (115) the latter is $\mathcal{O}(n^2)$, formula (117) is equivalent to $\varepsilon n^7 \ll \varepsilon^{1/2} n^2$, or $\varepsilon \ll n^{-10}$.

The restrictions in applying small- ε asymptotics are so severe that often problem (8) is more accurately approximated by treating ε as a *large* parameter: i.e., regarding (8) as a perturbation of a fourth-order equation by a second-order operator. This point of view may be developed systematically to explain the straight-line behavior in the range $10^{-2} < \varepsilon < 1$ in Figure 2 and predict that this behavior will continue indefinitely as ε increases.

8 Acknowledgements

We are grateful to Avshalom Manela for conversations introducing us to the flag problem, and to Manuel Kindelan for the suggestion that (41) was better solved as a boundary value problem than with shooting methods. DGS is grateful to Universidad Carlos III de Madrid and Banco de Santander for generous sabbatical support during the academic year 2009-10.

References

- [AM05] M. Argentina and L. Mahadevan, *Fluid-flow induced flutter of a flag*, Proceedings of the National Academy of Sciences (USA) **102** (2005), 1829–1834.
- [AS64] M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions*, 55, U.S. National Bureau of Standards, Applied Math., 1964.
- [Dow87] A. P. Dowling, *The dynamics of towed flexible cylinders*, Journal of Fluid Mechanics **187** (1987), 507–532.
- [Hin91] E. J. Hinch, *Perturbation methods*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 1991.
- [MH09] A. Manela and M.S. Howe, *On the stability and sound of an unforced flag*, Journal of Sound and Vibration **321** (2009), no. 3–5, 994–1006.

A Properties of Special Functions

The properties of the Bessel and Airy functions in this Appendix are taken from Chapters 9 and 10 of [AS64], respectively.

A.1 The Airy Functions, Ai and Bi

The Airy functions, Ai and Bi, are the standard solutions to the Airy equation $u'' - xu = 0$. The Wronskian of these solutions is constant:

$$\text{Ai}(x)\text{Bi}'(x) - \text{Ai}'(x)\text{Bi}(x) = \frac{1}{\pi}. \quad (118)$$

The Airy functions have exponential behavior for large positive x :

$$\text{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}, \quad \text{Bi}(x) \sim \frac{e^{\frac{2}{3}x^{3/2}}}{\sqrt{\pi}x^{1/4}}. \quad (119)$$

The definite integral of Ai converges and is known explicitly [AS64, 10.4.82]:

$$\int_0^\infty \text{Ai}(x) dx = \frac{1}{3}. \quad (120)$$

Incidentally,

$$\text{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}. \quad (121)$$

A.2 Bessel Functions of Order Zero

The two standard solutions to the zeroth-order Bessel Equation $u'' + \frac{1}{x}u' + u = 0$ are denoted J_0 and Y_0 . Despite the singularity of Bessel's equation at $x = 0$, $J_0(x)$ is differentiable near zero and in fact its power series converges for all x . By contrast, as $x \rightarrow 0$,

$$Y_0(x) = \frac{2}{\pi} \left(\log\left(\frac{1}{2}x\right) + \gamma \right) + \mathcal{O}(x^2 \log x)$$

where $\gamma = 0.577215665\dots$ is Euler's constant [AS64, 9.1.13]. For large x , these functions have the asymptotic approximation

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4), \quad (122)$$

$$Y_0(x) \sim \sqrt{\frac{2}{\pi x}} \sin(x - \pi/4) \quad (123)$$

In particular, J_0 has an infinite sequence of positive real zeros, beginning with

$$2.40482, 5.52007, 8.65372, \dots \quad (124)$$

In the paper we need linear combinations of Bessel functions with a slightly different argument as defined in (29) and (30). The Wronskian of \tilde{J} and \tilde{Y} may be computed from the fact that the Wronskian of J_0 and Y_0 equals $1/\pi x$ and making the substitution $x = 2\sqrt{\lambda_0 y}$:

$$\mathcal{W} = \tilde{J}(y)\tilde{Y}'(y) - \tilde{Y}(y)\tilde{J}'(y) = \frac{1}{y}. \quad (125)$$

\tilde{J} has the power series representation

$$\tilde{J}(y) = 1 - \lambda_0 y + \left(\frac{\lambda_0 y}{2!}\right)^2 - \left(\frac{\lambda_0 y}{3!}\right)^3 + \dots \quad (126)$$

This may be obtained by substitution into the power series for $J_0(x)$ or by computing coefficients recursively from the ODE (14).

B Proof: The Boundary-Layer Function Ψ

Proof of Lemma 3: (Existence) Equation (41) has an irregular singular point at $X = \infty$ but is regular everywhere else. Near the singular point, this equation admits a formal series solution

$$\Psi(X) \sim \log X + \sum_{k=1}^{\infty} c_k X^{-3k} \quad (127)$$

where $c_1 = 2/3$ and subsequent coefficients may be determined recursively. Therefore there is a solution of (41), an entire function of X , that has the asymptotic series (127) as $X \rightarrow \infty$, and in particular, the asymptotic behavior in the lemma. By subtracting off a multiples of $U^{(1)}, U^{(2)}$ we can also satisfy the boundary conditions at $X = 0$ without affecting the logarithmic behavior as $X \rightarrow \infty$.

(Uniqueness) Now suppose $\tilde{\Psi}$ is another solution of (41) that also satisfies the conditions of the lemma, and let $\Phi = \Psi - \tilde{\Psi}$. Then Φ solves (36) and $\Phi(X) = \mathcal{O}(1)$ as $X \rightarrow \infty$. Φ must be a linear combination of $\{U^{(i)}, i = 1, 2, 3, 4\}$, where these functions are defined in Subsection 4.4. However, $U^{(3)}$ and $U^{(4)}$ both grow more rapidly than $\mathcal{O}(1)$, and so does any nontrivial linear combination of them. We conclude that $\Phi = c_1 U^{(1)} + c_2 U^{(2)}$ for some constants c_1, c_2 , and since $\Phi(0) = \Phi'(0) = 0$, we find that $c_1 = c_2 = 0$. \square

Figure 6 shows a numerically computed graph of $\Psi(X)$. It was found by applying a boundary-value solver (Matlab's `bvp5c`) to (41), specifying two boundary conditions at $X = 0$ and an estimate for the first derivative at $X = 20$ obtained from the first 6 terms in (127).

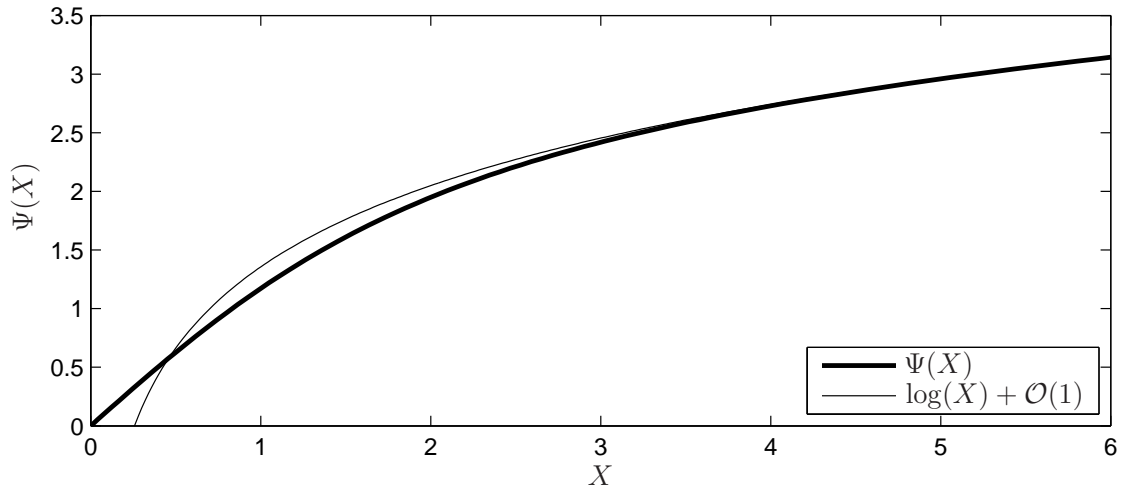


Figure 6: A plot of $\Psi(X)$. $\Psi(X)$ is the solution to (41) that has $\Psi(0) = \Psi''(0) = 0$ and a $\log(X) + \mathcal{O}(1)$ behavior for large X . Numerically, we find that $\Psi(X) - \log(X) \approx 1.3556$ at large X , so we also plot $\log(X) + 1.3556$ to illustrate the convergence of $\Psi(X)$ onto it.